Vodoniku sličan jon

- Rešava se $\hat{V}$ (problem) u sfernim koordinatama, pri čemu su $\hat{H}, \hat{\vec{L}}^2, \hat{L}_z$ PSKO - za česticu bez spina.

Sv. f-ja $\psi_{nlm}(\vec{r}) = C\, R_{nl}(r)\, Y_l^m(\theta, \varphi)$

radijalni deo

sferni harmonici

C - konstanta normiranja

$$\int \psi^*_{nlm}(\vec{r})\, \psi_{n'l'm'}(\vec{r})\, d^3\vec{r} = \delta_{nn'}\delta_{ll'}\delta_{mm'}$$

- Sv vrednosti energije su date izrazom

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}, \quad n = 1, 2, \dots$$

μ - relativna masa relativna čestice.

- Za datu vrednost kv. broja n, l uzima vrednosti iz skupa $l = 0, 1, 2, \dots n-1$ a kvantni broj m_l po pravila teorije ugaonog momenta.

- Rekurentne relacije koje mogu biti od interesa

$$\cos\theta\, Y_l^m = \sqrt{\frac{(l+|m|+1)(l-|m|+1)}{(2l+1)(2l+3)}}\, Y_{l+1}^m +$$

$$+ \sqrt{\frac{(l-|m|)(l+|m|)}{(2l-1)(2l+1)}}\, Y_{l-1}^m$$

$\longrightarrow$

1. Naći degeneraciju energetskog nivoa elektrona u vodoniku sličnom jonu, koja odgovara glavnom kvantnom broju n:

a) Za česticu bez spina

b) Za spin $s = \frac{1}{2}$

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a) Za fiksirano n : $l = 0, 1, \dots n-1$

Za fiksirano l : $m_l = -l, \dots 0, \dots l$

$$\text{Degeneracija } g_n = \sum_{l=0}^{n-1} (2l+1) = 2\sum_{l=0}^{n-1} l + \sum_{l=0}^{n-1} 1 =$$

$$= 2\,\frac{(n-1)(n-1+1)}{2} + n = n^2 - n + n = n^2$$

Primer $n=2$ (fiksirano)

$l = 0, 1 \Rightarrow$ $l = 0$ $m_l = 0$

$l = 1$ $m_l = -1, 0, 1$

$|n\ l\ m_l\rangle$

2	0	0
2	1	-1
2	1	0
2	1	1

} ukupno $4 = (n=2)^2$

b) bez spina $\rightarrow$ ima n^2 stanja

sa spinom $\rightarrow m_s = \pm\frac{1}{2}$ (2 orijentacije) u $2n^2$ sta[nja]

$|n\ l\ m_l\rangle \otimes |m_s\rangle$ (v. gornji primer)

2. Dipolni moment vodoniku sličnog jona definisan je kao $\vec{p}=e\vec{r}$, gde je $\vec{r}$ vektor relativnog položaja "valentnog" elektrona u odnosu na atomsko jezgro. Naći očekivanu vrednost ove veličine u stanjima

a) $|\psi_{100}\rangle$

b) $|\psi_{200}\rangle$

za domaći

c) $|\psi_{210}\rangle$

d) $|\psi_{211}\rangle$

e) $|\psi_{21-1}\rangle$

za ispit

$$\langle \hat{\vec{p}} \rangle = e\langle \hat{\vec{r}} \rangle \Rightarrow$$

$$\langle \hat{p}_i \rangle = e\langle \hat{x}_i \rangle, \quad i = x, y, z$$

$$\langle \hat{x} \rangle = \int \psi^*_{100}(\vec{r})\, x\, \psi_{100}(\vec{r})\, d^3\vec{r}$$

$$\psi_{nlm}(\vec{r}) = R_{nl}(r)\, Y_l^m(\theta, \varphi)$$

$$R_{nl}:\quad R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}}$$

$$R_{20}(r) = 2\left(\frac{Z}{2a_0}\right)^{3/2}\left[1 - \frac{Zr}{2a_0}\right] e^{-\frac{Zr}{2a_0}}$$

$$R_{21}(r) = \sqrt{\frac{1}{3}}\left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{a_0}\, e^{-\frac{Zr}{2a_0}}$$

$$Y_m^l(\theta, \varphi)$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^{\pm 1} = \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta \, e^{\pm i\varphi}$$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}} \left(2\cos^2\theta - \sin^2\theta\right)$$

$$Y_2^{\pm 1} = \mp \frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta\cos\theta \, e^{\pm i\varphi}$$

$$Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{\pm 2i\varphi}$$

$$\langle \hat{x} \rangle = \int R_{10}(r)\, Y_0^0(\theta,\varphi) \; x \; R_{10}(r)\, Y_0^0(\theta,\varphi)\, d^3\vec{r}$$

prelazimo na sferne koordinate

$$\langle \hat{x} \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{4\left(\frac{Z}{a_0}\right)^3 e^{-\frac{2rZ}{a_0}}}{4\pi} (r\sin\theta\cos\varphi)\, r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 \int_0^{+\infty} r^3 e^{-\frac{2rZ}{a_0}} dr \int_0^\pi \sin^2\theta \, d\theta \int_0^{2\pi} \cos\varphi \, d\varphi$$

$$F \quad \int_0^\infty r^3 e^{-\frac{2rZ}{a_0}} dr =$$

$$\left(\frac{2rZ}{a_0} = t \Rightarrow r = \frac{a_0}{2Z} t \Rightarrow dr = \frac{a_0}{2Z} dt \right)$$

$$= \int_0^{+\infty} \left(\frac{a_0}{2Z}\right)^3 t^3 e^{-t} \frac{a_0}{2Z} dt = \left(\frac{a_0}{2Z}\right)^4 \underbrace{\int_0^{+\infty} t^3 e^{-t} dt}_{\Gamma(4) = 3!} =$$

$$= 3!\left(\frac{a_0}{2Z}\right)^4$$

$$\int_0^{\pi} \sin^2\theta\, d\theta = \int_0^{\pi} \frac{1-\cos 2\theta}{2} d\theta = \frac{1}{2}\int_0^{\pi} d\theta - \frac{1}{2}\int_0^{\pi} \cos 2\theta\, d\theta$$

$$= \frac{\pi}{2} - \frac{1}{4}\int_0^{2\pi} \cos 2\theta\, d(2\theta) = \frac{\pi}{2} - \frac{1}{4}\sin 2\theta \Big|_0^{2\pi} = \frac{\pi}{2}$$

$$\int_0^{2\pi} \cos\varphi\, d\varphi = \sin\varphi \Big|_0^{2\pi} = 0$$

$$\boxed{\langle \hat{x} \rangle = 0}$$

$$\langle \hat{y} \rangle = \int R_{10}(r)\, Y_1^0(\theta,\varphi)\, \hat{y}\, R_{10}(r)\, Y_1^0(\theta,\varphi)\, d^3\vec{r}$$

$$= \iiint R_{10}^2(r)(Y_0^0)^2\, r\cos\theta\cos\varphi \quad r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int_0^{\infty} R_{10}^2(r)\, r^3\, dr \int_0^{\pi} \frac{\cos\theta\sin\theta\, d\theta}{4\pi} \int_0^{2\pi} \cancel{\cos\varphi\, d\varphi} = 0$$

$$\boxed{\langle \hat{y} \rangle = 0}$$

$$\langle \hat{z} \rangle = \iiint R_{10}^{(2)}(r)(Y_0^0)^2\, r\cos\theta \quad r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \underbrace{\int_0^{+\infty} R_{10}^2(r)\, r^2\, dr}_{\text{Izracunato}} \int_0^{\pi} \frac{\cos\theta\sin\theta}{4\pi} d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{1}{2}\, 3!\left(\frac{a_0}{2z}\right)^4 \int_0^{\pi} \cos\theta\sin\theta\, d\theta$$

$$= 3\left(\frac{a_0}{2z}\right)^4 \frac{1}{2}\int_0^{\pi} 2\sin\theta\cos\theta\, d\theta$$

$$= \frac{3}{2}\left(\frac{a_0}{2z}\right)^4 \int_0^{\pi} \sin 2\theta\, d\theta = \frac{3}{2}\left(\frac{a_0}{2z}\right)^4 \frac{1}{2}\int_0^{\pi} \sin 2\theta\, d(2\theta)$$

$$= \frac{3}{4}\left(\frac{a_0}{2z}\right)^4 \left(-\cos 2\theta\right)\Big|_0^{\pi} = 0$$

Dakle $\langle \vec{P} \rangle = 0$

3. Naći očekivanu vrednost intenziteta dipolnog momenta definisanog u prethodnom zadatku, u stanjima takođe tamo zadatim.

$$\hat{\vec{P}} = e\hat{\vec{r}}$$

~~a) za domaći~~

a) se radi na vežbama, a pod b) za vežbanje

$$|\hat{\vec{P}}| = e|\hat{\vec{r}}| \xrightarrow[\text{koord.}]{\text{sferne}} e\hat{r}$$

b) $|\psi_{200}\rangle \longrightarrow \quad \psi_{200}(\vec{r}) = R_{20}(r)\, Y_0^0(\theta,\varphi)$

$$= \frac{1}{\sqrt{4\pi}}\, 2\left(\frac{Z}{2a_0}\right)^{3/2}\left[1-\frac{Zr}{2a_0}\right]e^{-\frac{Zr}{2a_0}}$$

$$\langle|\hat{P}|\rangle = \langle\psi_{200}|\, e|\hat{\vec{r}}|\, |\psi_{200}\rangle = e\langle\psi_{200}|\,|\hat{\vec{r}}|\,|\psi_{200}\rangle$$

$$= e\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\frac{1}{\sqrt{4\pi}}\, 2\left(\frac{Z}{2a_0}\right)^{3/2}\left[1-\frac{Zr}{2a_0}\right]e^{-\frac{Zr}{2a_0}}\; r\,\frac{2}{\sqrt{4\pi}} \otimes$$

$$\otimes \left(\frac{Z}{2a_0}\right)^{3/2}\left[1-\frac{Zr}{2a_0}\right]e^{-\frac{Zr}{2a_0}}\; r^2\sin\theta\, d\theta\, dr\, d\varphi$$

$$= \frac{e\,\cancel{4}}{4\pi}\left(\frac{Z}{2a_0}\right)^3\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi} r^3\left[1-\frac{Zr}{2a_0}\right]^2 e^{-\frac{Zr}{a_0}}\sin\theta\, d\theta\, dr\, d\varphi$$

$$= \frac{1}{\pi}\left(\frac{Z}{2a_0}\right)^3 e\int_0^{+\infty} r^3\left[1-\frac{Zr}{2a_0}\right]^2 e^{-\frac{Zr}{a_0}}dr\,\underbrace{\int_0^{\pi}\sin\theta\, d\theta}_{2}\;\underbrace{\int_0^{2\pi}d\varphi}_{2\pi}$$

$$= \frac{4\cancel{\pi}}{\cancel{\pi}}\left(\frac{Z}{2a_0}\right)^3 e\,\underbrace{\int_0^{+\infty} r^3\left[1-\frac{Zr}{2a_0}\right]^2 e^{-\frac{Zr}{a_0}}dr}_{I}$$

$$\langle |\hat{\vec{p}}| \rangle = 4 \left(\frac{Z}{2a_0}\right)^{3} I$$

$$I = \int_0^{+\infty} r^3 \left(1 - \frac{Zr}{a_0} + \frac{Z^2 r^2}{4a_0^2}\right) e^{-\frac{Zr}{a_0}}\, dr$$

$$= \underbrace{\int_0^{+\infty} r^3 e^{-Zr/a_0}\, dr}_{I_1} - \frac{Z}{a_0} \underbrace{\int_0^{+\infty} r^4 e^{-\frac{Zr}{a_0}}\, dr}_{I_2} + \frac{Z^2}{4a_0^2} \underbrace{\int_0^{+\infty} r^5 e^{-\frac{Zr}{a_0}}\, dr}_{I_3}$$

smena I_1, I_2, I_3

$$\frac{Zr}{a_0} = t \Rightarrow r = \frac{a_0}{Z} t \Rightarrow dr = \frac{a_0}{Z} dt$$

$$= \left(\frac{a_0}{Z}\right)^4 \int_0^{+\infty} t^3 e^{-t} dt - \frac{\cancel{Z}}{a_0} \left(\frac{a_0}{Z}\right)^5 \int_0^{+\infty} t^4 e^{-t} dt +$$

$$+ \frac{Z^2}{4a_0^2} \left(\frac{a_0}{Z}\right)^6 \int_0^{+\infty} t^5 e^{-t} dt$$

$$= \left(\frac{a_0}{Z}\right)^4 \left[\underbrace{\int_0^{+\infty} t^3 e^{-t} dt}_{\Gamma(4)} - \underbrace{\int_0^{+\infty} t^4 e^{-t} dt}_{\Gamma(5)} + \frac{1}{4} \underbrace{\int_0^{+\infty} t^5 e^{-t} dt}_{\Gamma(6)} \right]$$

$$= \left(\frac{a_0}{Z}\right)^4 \left[\Gamma(4) - \Gamma(5) + \frac{1}{4} \Gamma(6) \right]$$

$$= \left(\frac{a_0}{Z}\right)^4 \left[3! - 4! + \frac{5!}{4} \right] = \left(\frac{a_0}{Z}\right)^4 \left[6 - 24 + 30 \right]$$

$$= 12 \left(\frac{a_0}{Z}\right)^4$$

$$\langle |\hat{\vec{p}}| \rangle = e\,4 \left(\frac{Z}{2a_0}\right)^3 12 \left(\frac{a_0}{Z}\right)^4 = \cancel{48}\,\frac{a_0}{\cancel{Z}}$$

$$= 6e\,\frac{a_0}{Z}$$

Primedba: Ostalo za ispit!

4. Naći očekivanu vrednost X-koordinate vektora položaja u stanju vodonikovog atoma $|\psi_{211}\rangle$

Smer A ovaj zadatak za vezbu

$$\langle \hat{x} \rangle = \langle \psi_{211} | \hat{x} | \psi_{211} \rangle$$

$$= \int \psi^*_{211}(\vec{r})\, x\, \psi_{211}(\vec{r})\, d^3\vec{r} \xrightarrow{\text{Sferne koordinate}}$$

$$\psi_{211}(\vec{r}) = R_{21}(r)\, Y_1^1(\theta,\varphi)$$

$$R_{21}(\vec{r}) = \sqrt{\frac{1}{3}} \left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-\frac{Zr}{2a_0}}$$

$$Y_1^1(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta\, e^{i\varphi}$$

$$\psi_{211}(\vec{r}) = -\frac{1}{2\sqrt{2\pi}} \left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{a_0} \sin\theta\, e^{-\frac{Zr}{2a_0}} e^{i\varphi}$$

$$|\psi_{211}(\vec{r})|^2 = \frac{1}{8\pi^2}\left(\frac{Z}{2a_0}\right)^3 \left(\frac{Zr}{a_0}\right)^2 \sin^2\theta\, e^{-\frac{Zr}{a_0}}$$

$$\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi} \underbrace{\frac{1}{8\pi^2}\left(\frac{Z}{2a_0}\right)^3 \left(\frac{Zr}{a_0}\right)^2 \sin^2\theta\, e^{-\frac{Zr}{a_0}}}_{|\psi_{211}(\vec{r})|^2} \underbrace{r\sin\theta\cos\varphi}_{x} \underbrace{r^2 \sin\theta\, dr\, d\theta\, d\varphi}_{d^3\vec{r}}$$

$$\frac{1}{8\pi^2}\left(\frac{Z}{2a_0}\right)^3\left(\frac{Z}{a_0}\right)^5\int_0^{+\infty} r^5 e^{-\frac{Zr}{a_0}}dr \int_0^{\pi} \sin^4\theta \, d\theta \underbrace{\int_0^{2\pi} \cos\varphi \, d\varphi}_{0} = 0$$

Za domaći $\langle \hat{y} \rangle$ i $\langle \hat{z} \rangle$

$$\boxed{y = r\sin\theta\sin\varphi}$$

Verovatno je potrebno koristiti rekurentne relacije? (NE!)

Domaći: Naći očekivanu vrednost opservable $\hat{X}\hat{Y}$ u stanju $|\psi_{211}\rangle$

7. Stanje elektrona u vodonikovom atomu zadato je uslovima: Verovatnoća da se merenjem energije u ovom stanju dobije vrednost $E_n = -\mu Z^2 e^4 / 8\hbar^2$ (μ - relativna masa) jednaka je jedinici. Pri tome, verovatnoće da se simultanim merenjem dobiju i vrednosti $2\hbar^2$ za $\vec{L}^2$ i $\pm\hbar$ za $\hat{L}_z$ iznose nula. Odrediti takvo stanje i naći disperziju opservable $\hat{r}^n$ u tom stanju.

Svojstvene vrednosti energije za elektron

$$\left.\begin{array}{l} E_n = -\dfrac{\mu Z^2 e^4}{2\hbar^2 n^2} \\ \\ E_n = -\dfrac{\mu e^4}{8\hbar^2} \end{array}\right\} \Rightarrow \quad n^2 = 4 \Rightarrow n = 2$$

Ako je $n = 2, \Rightarrow l = 0, 1$

$m_l = 0 \qquad m_l = -1, 0, 1$

$|\psi_{200}\rangle \quad |\psi_{210}\rangle \quad |\psi_{211}\rangle \quad |\psi_{21-1}\rangle$

Može biti bilo koje od ovih stanja, ali može i svaka njihova superpozicija.

$$|\psi\rangle = \sum_{l=0}^{1} \sum_{m_l=-l}^{l} c_{l m_l} |\psi_{2 l m_l}\rangle$$

~~$c_{lm_l} = |\langle\psi_{2lm_l}|\psi\rangle|^2$~~

$c_{\ell m_\ell} = \langle \psi_{2\ell m_\ell} | \psi \rangle$

$$W(\hat{\vec{L}}^2, \hat{L}_z, |\psi\rangle; \ell(\ell+1)\hbar^2, m_\ell\hbar) = \sum_{? \, \ell, m_\ell} |\langle \psi_{2\ell m_\ell} | \psi \rangle|^2$$

U zadatku je zadano

$$W(\hat{\vec{L}}^2, \hat{L}_z, |\psi\rangle, 1(1+1)\hbar^2, \hbar) = |c_{11}|^2 = 0 \Rightarrow c_{11} = 0$$

$$W(\hat{\vec{L}}, \hat{L}_z, |\psi\rangle, 1(1+1)\hbar^2, -\hbar) = |c_{1-1}|^2 = 0 \Rightarrow c_{1-1} = 0$$

$$|\psi\rangle = c_{00}|\psi_{200}\rangle + c_{10}|\psi_{210}\rangle + \cancel{c_{1-1}|\psi_{21-1}\rangle} + \cancel{c_{11}|\psi_{211}\rangle}$$

$$|\psi\rangle = c_{00}|\psi_{200}\rangle + c_{10}|\psi_{210}\rangle$$

Potrebni su dodatni uslovi da bi se odredili koeficijenti c_{00} i c_{10} $\left(|c_{00}|^2 + |c_{10}|^2 = 1\right)$

$$\langle \hat{r}^n \rangle = c_{00}\langle \psi_{200} | \hat{r}^n | \psi_{200} \rangle + c_{10}\langle \psi_{210} | \hat{r}^n | \psi_{210} \rangle$$

$$\langle \psi_{200} | \hat{r}^n | \psi_{200} \rangle = \int \psi^*_{200}(\vec{r})\, \hat{r}^n\, \psi_{200}(\vec{r})\, d^3\vec{r}$$

$$\psi_{200}(\vec{r}) = R_{20}(r)\, Y^0_0(\theta, \varphi)$$

$$R_{20}(r) = 2\left(\frac{Z}{2a_0}\right)^{3/2}\left[1 - \frac{Zr}{2a_0}\right] e^{-\frac{Zr}{2a_0}}$$

$$Y^0_0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$$

$$\langle\psi_{200}|\hat{r}^n|\psi_{200}\rangle = 4\left(\frac{Z}{2a_0}\right)^3 \cdot \frac{1}{4\pi}\int_0^{+\infty} r^n\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}}\, r^2\, dr \cdot \underbrace{\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi}_{4\pi}$$

$$= 4\left(\frac{Z}{2a_0}\right)^3 \int_0^{+\infty} r^{n+2}\left(1-\frac{Zr}{2a_0}\right)^2 e^{-\frac{Zr}{a_0}}\, dr$$

$$= \frac{Z^3}{2a_0^3}\left(I_1 + I_2 + I_3\right)$$

$$I_1 = \left(\frac{a_0}{Z}\right)^{n+3}\Gamma(n+3)$$

$$I_2 = -\frac{Z}{a_0}\left(\frac{a_0}{Z}\right)^{n+4}\Gamma(n+4)$$

$$I_3 = \left(\frac{Z}{2a_0}\right)^2\left(\frac{a_0}{Z}\right)^{n+5}\Gamma(n+5)$$

Analogno se računa

$$\langle\psi_{210}|\hat{r}^n|\psi_{210}\rangle = \ldots$$

Završiti za domaći

6. Naći najverovatniju vrednost azimutskog ugla θ m elektrona u vodonikovom atomu, ako je zadata vrednost orbitalnog kvantnog broja $l=1$, $\boxed{\text{za svako } m_e \text{ posebno.}}$ Ispit
i $m=0$.

$$W\left(r\in[r_0, r_0+dr], \theta\in[\theta_0, \theta_0+d\theta], \varphi\in[\varphi_0, \varphi_0+d\varphi]\right) \overset{d}{=}$$

$$= |\psi(r,\theta,\varphi)^{*}|^2 r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$g_v(r,\theta,\varphi) \overset{d}{=} |\psi(r,\theta,\varphi)|^2 r^2 \sin\theta$$

$$g_v(\theta) = \int_0^{\infty}\int_0^{2\pi} g_v(r,\theta,\varphi)\, dr\, d\varphi$$

$$l=1 \Rightarrow m_e = 1, 0, -1$$

$$\psi(r,\theta,\varphi) = R_{nl}(r) Y_l^m(\theta,\varphi)$$

$$\psi_{n1m_e} = R_{n1}(r) Y_1^m(\theta,\varphi)$$

$$g_v^m(\theta) = \int_0^{\infty}\int_0^{2\pi} R_{n1}^2(r) \left|Y_1^m(\theta,\varphi)\right|^2 r^2 \sin\theta \, dr\, d\varphi$$

Rešavamo za $m=0$, ostalo za domaći

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta \qquad Y_1^{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$g_v^{(0)}(\theta) = \int_0^{\infty}\int_0^{2\pi} R_{n1}^2(r)\ \frac{1}{4}\frac{3}{\pi}\cos^2\theta\ \ r^2\sin\theta\, dr\, d\varphi$$

$$= \frac{3}{4\pi}\underbrace{\int_0^{+\infty} R_{n1}^2(r)\, r^2\, dr}_{C_r}\ \int_0^{2\pi} d\varphi\ \ \sin\theta\cos^2\theta$$

$$= \frac{3\cdot 2\pi}{4\pi}\cdot C_r\ \sin\theta\cos^2\theta$$

$$= c\ \sin\theta\cos^2\theta$$

$$\frac{dg_v^{(0)}(\theta)}{d\theta} = 0 \Rightarrow \theta_{max}$$

$$\frac{dg_v^{(0)}(\theta)}{d\theta} = c\left[\cos\theta\cos^2\theta + \sin\theta\ 2\cos\theta(-\sin\theta)\right]$$

$$= c\left[\cos^3\theta - 2\sin^2\theta\cos\theta\right]$$

$$\frac{dg_v^{(0)}(\theta)}{d\theta} = 0 \Rightarrow \left[\cos^3\theta - 2\sin^2\theta\cos\theta\right] = 0$$

$$\cos\theta\left(\cos^2\theta - 2\sin^2\theta\right) = 0$$

$$\cos\theta\left(\underbrace{\cos^2\theta - \sin^2\theta}_{} - \sin^2\theta\right) = 0$$

$$\cos\theta\left(\cos 2\theta - \frac{1-\cos 2\theta}{2}\right) = 0$$

$$\cos\theta\left(\cos 2\theta + \frac{1}{2}\cos 2\theta - \frac{1}{2}\right) = 0$$

$$\cos\theta \left(\frac{3}{2}\cos 2\theta - \frac{1}{2} \right) = 0$$

$$\underline{\theta} = \frac{\pi}{2} \quad \wedge \quad \theta = \frac{1}{2} \arccos \frac{1}{3}$$

I) Elektron u vodonikovom atomu se nalazi u stanju $|\psi\rangle = f(r)\,|j=\frac{3}{2}\ m_j=\frac{1}{2}\rangle$. Korišćenjem tehnike Klebš-Gordanovih koeficijenata naći verovatnoću da se simultanim merenjem kvadrata momenta impulsa, z-projekcije momenta impulsa i spina dobiju vrednosti $2\hbar^2$, 0 i $\hbar/2$, redom. Zanemariti radijalni deo prostora stanja.

$$W\left(\hat{\vec{L}}^2, \hat{L}_z, \hat{S}_z; |\psi\rangle; 2\hbar^2, 0, \frac{\hbar}{2}\right) =$$

$$= \langle\psi|\, \hat{I}_r \otimes \hat{P}_\Omega^{(lm)} \otimes \hat{\Pi}_S^{(sm)}\, |\psi\rangle$$

$$l(l+1)\hbar^2 = 2\hbar^2 \Rightarrow l = 1$$

$$m_l = 0$$

$$m_s = \frac{1}{2}$$

$$\hat{P}_\Omega^{(lm)} = |lm\rangle\langle lm| \qquad \hat{\Pi}_S^{(sm)} = |sm\rangle\langle sm|$$

$$W(\dots) = \langle\psi|\, \hat{I}_r \otimes |10\rangle\langle 10| \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle\langle\tfrac{1}{2}\tfrac{1}{2}|\psi\rangle$$

$$= \left|\langle\psi|10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle\right|^2$$

$$= \int_0^{\infty} |f(r)|^2 r^2 dr \left|\langle j=\tfrac{3}{2}\ m_j=\tfrac{1}{2}|10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle\right|^2$$

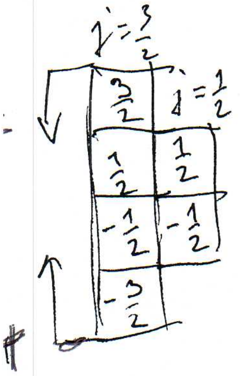
l $\qquad$ s

$|11\rangle \qquad |\tfrac{1}{2}\tfrac{1}{2}\rangle \qquad j = l + s$

$|10\rangle \qquad \tfrac{3}{2} = 1 + s \Rightarrow s = \tfrac{1}{2}$

$|1{-1}\rangle \qquad |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle$

$$|j=\tfrac{3}{2}\, m_j=\tfrac{1}{2}\rangle = c_1 |10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + c_2 |11\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle$$



$$|c_1|^2 + |c_2|^2 = 1$$

$$|j=\tfrac{3}{2};\, m_j=\tfrac{3}{2}\rangle = |11\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$\hat{J}_- |\tfrac{3}{2}\tfrac{3}{2}\rangle = \hbar \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}-1)}\ |\tfrac{3}{2}\tfrac{1}{2}\rangle$$

$$= \sqrt{3}\hbar\ |\tfrac{3}{2}\tfrac{1}{2}\rangle \quad (A.C.)$$

$$\hat{J}_- = \hat{L}_- \otimes \hat{I}_s + \hat{I}_L \otimes \hat{S}_-$$

$$(\hat{L}_- \otimes \hat{I}_s + \hat{I}_L \otimes \hat{S}_-)\, |11\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle = \hat{L}_- |11\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + |11\rangle \otimes \hat{S}_- |\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= \hbar \sqrt{1(1+1) - 1(1-1)}\ |10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + |11\rangle \otimes \hbar \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}-1)}\ |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle$$

$$= \sqrt{2}\hbar\ |10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + \hbar\ |11\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle \quad (\Delta.C)$$

$$A.C. = \Delta.C.$$

$$\sqrt{3}\hbar\ |\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{2}\hbar\ |10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + \hbar\ |11\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle$$

$$|\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\ |10\rangle \otimes |\tfrac{1}{2}\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\ |11\rangle \otimes |\tfrac{1}{2}{-\tfrac{1}{2}}\rangle$$

Jasno je da je $W(\ldots) = \frac{2}{3}$

8. Dato je stanje elektrona u vodonikovom atomu

$$\psi(\vec{r}) = (2\pi a_0^3)^{-1/2} e^{-r/a_0} + (128 a_0^5)^{-1/2} r e^{-r/2a_0} (\cos\theta - \sin\theta e^{i\varphi})$$

Naći verovatnoću da se merenjem energije u trenutku t dobiju vrednosti koje odgovaraju kvantnom broju $n=1,2$. Kolika je verovatnoća da se merenjem dobije rezultat $n>2$?

Elektron je u stanju $|\psi\rangle$. U kasnijem trenutku biće u stanju $\hat{U}(t)|\psi\rangle = |\psi(t)\rangle$.

Verovatnoća u kasnijem trenutku

$$W(\hat{H}, E_n, |\psi(t)\rangle) = \langle\psi(t)|\hat{P}_n|\psi(t)\rangle$$

a za konzervativne sisteme je jednaka verovatnoći u $t=0$, tj. $|\psi(t)\rangle = |\psi\rangle$

$$W(\hat{H}, E_n, |\psi(t)\rangle) = \langle\psi|\hat{P}_n|\psi\rangle$$

Za $n=1$, $\hat{P}_1 = ?$

$n=1 \Rightarrow l=0 \Rightarrow m=0 \Rightarrow \hat{P}_1 = |100\rangle\langle 100|$

$$W(\dots) = |\langle 100|\psi\rangle|^2$$

U koordinatnoj reprezentaciji

$$\langle 100|\psi\rangle = \int \psi^{*}_{100}(\vec{r})\,\psi(\vec{r})\,d^3\vec{r}$$

$$= \int R_{10}(r)\,Y_0^0(\theta,\varphi)\,\psi(\vec{r})\,r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{\frac{3}{2}} e^{-\frac{Zr}{a_0}} \;;\; Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$= \mu\, e^{-\frac{Zr}{a_0}}$$

$$\psi(\vec{r}) = \alpha\, e^{-\frac{r}{a_0}} + \beta r\, e^{-\frac{r}{2a_0}}\left(\cos\theta - \sin\theta\, e^{i\varphi}\right)$$

$$\Downarrow$$

$$\boxed{R_{10}(r)\,\psi(\vec{r})} = \alpha\mu\, e^{-\frac{r}{a_0}(Z+1)} + \mu\beta r\, e^{-\frac{r}{a_0}(2Z+1)}\cos\theta$$

$$- \mu\beta r\, e^{-\frac{r}{a_0}(2Z+1)}\sin\theta\, e^{i\varphi}$$

Dando

$$\langle 100|\psi\rangle = \frac{\alpha\beta}{\sqrt{4\pi}} \int e^{-\frac{r}{a_0}(Z+1)}\, r^2\sin\theta\,dr\,d\theta\,d\varphi \;+$$

$$\frac{\mu\beta}{\sqrt{4\pi}} \int r^3 e^{-\frac{r}{a_0}(2Z+1)} \cos\theta\sin\theta\,dr\,d\theta\,d\varphi \;-$$

$$- \frac{\mu\beta}{\sqrt{4\pi}} \int r^3 e^{-\frac{r}{a_0}(2Z+1)}\, e^{i\varphi}\sin^2\theta\,dr\,d\theta\,d\varphi$$

$$I_1 = \int e^{-\frac{r}{a_0}(Z+1)} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$= \int_0^{+\infty} e^{-\frac{r}{a_0}(Z+1)} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= 4\pi \int_0^{+\infty} e^{-\frac{r}{a_0}(Z+1)} r^2 dr$$

Smena $\frac{r}{a_0}(Z+1) = t$

$$r = \frac{a_0 t}{Z+1} \Rightarrow dr = \frac{a_0}{Z+1} dt$$

$$= 4\pi \frac{a_0^3}{(Z+1)^3} \int_0^{+\infty} e^{-t} t^2 dt$$

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$$

$$= 4\pi \frac{a_0^3}{(Z+1)^3} \Gamma(3) = \frac{8\pi a_0^3}{(Z+1)^3}$$

$I_2 = 0$ jer $\int_0^{\pi} \cos\theta \sin\theta \, d\theta = 0$

$I_3 = 0$ jer $\int_0^{2\pi} e^{i\varphi} d\varphi = 0$

$$\langle 100|\Psi\rangle = \frac{\alpha}{\sqrt{4\pi}} \frac{8\pi a_0^3}{(Z+1)^3}$$

$$= \frac{(2\pi a_0^3)^{-\frac{1}{2}} \, 2\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}}{\sqrt{4\pi} \, (Z+1)^3} \cdot 8\pi a_0^3 = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2\pi a_0^3}} \frac{Z^{\frac{3}{2}}}{(Z+1)^3} \frac{1}{\sqrt{a_0^3}} 8\pi a_0^3$$

$$= \frac{1}{\pi} \frac{1}{\sqrt{2} a_0^3} \left(\frac{\sqrt{Z}}{Z+1}\right)^3 8\pi a_0^3$$

$$\langle 100|\psi\rangle = 4\sqrt{2}\left(\frac{\sqrt{Z}}{Z+1}\right)^3$$

Za $Z=1$

$$\langle 100|\psi\rangle = 4\sqrt{2}\,\frac{1}{8} = \frac{\sqrt{2}}{2} \Rightarrow$$

$$|\langle 100|\psi\rangle|^2 = \frac{1}{2}$$

Za $n=2$, $\hat{P}_2 = ?$

$n=2 \Rightarrow l=0,\ m_l=0$
$\qquad\qquad l=1,\ m_l=-1,0,1$

$$\hat{P}_2 = |200\rangle\langle 200| + |210\rangle\langle 210| + |211\rangle\langle 211| + |21{-1}\rangle\langle 21{-1}|$$

$$W(\dots) = \langle\psi|\hat{P}_2|\psi\rangle$$

Treba izračunati

$$\left.\begin{array}{l}\langle 200|\psi\rangle = ?\\ \langle 210|\psi\rangle = ?\\ \langle 211|\psi\rangle = ?\\ \langle 21{-1}|\psi\rangle = ?\end{array}\right\}\ \text{vezba}$$

$n > 2$

Isključivi događaji

$$W(n=1) + W(n=2) + W(n>2) = 1$$

$$W(n>2) = 1 - W(n=1) - W(n=2)$$

Naći najverovatnije r za stanja $\psi_{n\ell m}(\vec{r})$, $n=1,2$ i bilo koje ℓ i m_ℓ. (na vežbama za $\ell=0$, $m_\ell=0$)

$$g_r(r,\theta,\varphi) = |\psi_{n\ell m}(r,\theta,\varphi)|^2 r^2 \sin\theta$$

$$g_r(r) = r^2 \int_0^{\pi}\int_0^{2\pi} |\psi_{n\ell m}(r,\theta,\varphi)|^2 \sin\theta\, d\theta\, d\varphi$$

Za ispit, upr!

za $n=1$, $\ell=0$, $m_\ell=0$

$$\boxed{\psi_{100} = R_{10}(r)\, Y_0^0(\theta,\varphi) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}} \frac{1}{\sqrt{4\pi}}}$$

$$\underline{\underline{g_r(r)}} = \frac{4}{4\pi}\int_0^{\pi}\int_0^{2\pi}\left(\frac{Z}{a_0}\right)^3 e^{-\frac{2Zr}{a_0}} \sin\theta\, d\theta\, d\varphi \cdot r^2$$

ispušteno

$$= \frac{4}{\pi}\left(\frac{Z}{a_0}\right)^3 e^{-\frac{2Zr}{a_0}} \underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\varphi}_{4\pi} \cdot r^2$$

$$= \underline{\underline{4r^2\left(\frac{Z}{a_0}\right)^3 e^{-\frac{2Zr}{a_0}}}}$$

$$\frac{dg_r(r)}{dr} = 0 \Rightarrow r_m$$

Kada se izračuna $r_m = \frac{a_0}{Z}$

10 Naći disperziju opservabli $\hat{x}$, $\cancel{\hat{p}_x}$, $\hat{x}\hat{y}$, u stanjima a) $|\psi_{100}\rangle$, b) $|\psi_{210}\rangle$, koja se odnose na kasniji trenutak t. ispit

a)

$$\Delta \hat{A}_t = \sqrt{\langle\psi(t)|\hat{A}^2|\psi(t)\rangle - (\langle\psi(t)|\hat{A}|\psi(t)\rangle)^2}$$

$$\hat{U}(t)|\psi\rangle = |\psi(t)\rangle \Rightarrow \quad \langle\psi(t)| = \langle\psi|\hat{U}^+(t)$$

$$\langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi|\hat{U}^+(t)\hat{A}\hat{U}(t)|\psi\rangle$$

$$\langle\psi(t)|\hat{A}^2|\psi(t)\rangle = \langle\psi|\hat{U}^+(t)\hat{A}^2\hat{U}(t)|\psi\rangle$$

Za $|\psi_{100}\rangle$ važi

$$\hat{U}(t)|\psi_{100}\rangle = e^{-\frac{i}{\hbar}E_1 t}|\psi_{100}\rangle$$

$$\langle\psi_{100}(t)|\hat{A}|\psi_{100}(t)\rangle = \langle\psi_{100}|e^{\frac{i}{\hbar}E_1 t}\hat{A}\, e^{-\frac{i}{\hbar}E_1 t}|\psi_{100}\rangle$$

$$= \langle\psi_{100}|\hat{A}|\psi_{100}\rangle$$

$$\langle\psi_{100}(t)|\hat{A}^2|\psi_{100}(t)\rangle = \langle\psi_{100}|\hat{A}^2|\psi_{100}\rangle$$

$$\hat{A} = \hat{x}$$

$$\Delta\hat{x} \equiv \Delta\hat{x}_t = \sqrt{\langle\psi_{100}|\hat{x}^2|\psi_{100}\rangle - \langle\psi_{100}|\hat{x}|\psi_{100}\rangle^2}$$

$$\langle\psi_{100}|\hat{x}^2|\psi_{100}\rangle = \int \psi^*_{100}(\vec{r})\,\hat{x}^2\,\psi_{100}(\vec{r})\,d^3\vec{r}$$

$$= \int_0^{+\infty}\int_0^{2\pi}\int_0^{\pi} (r\sin\theta\cos\varphi)^2\, R_{10}^2(r)\left(Y_0^0(\theta,\varphi)\right)^2 r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{4\pi}\int_0^{+\infty} r^4 R_{10}^2(r)\,dr \int_0^{\pi}\sin^3\theta\, d\theta \int_0^{2\pi}\cos^2\varphi\, d\varphi$$

Završiti a ostalo za domaći (ispit)

HE

$$\langle\hat{p}_x\rangle = \int \psi_{nlm}(\vec{r})\left(-i\hbar\frac{\partial}{\partial x}\right)\psi_{nlm}(\vec{r})\,d^3\vec{r}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x}$$

$$x = x(r,\theta,\varphi)$$

$x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\theta$

$$\frac{\partial^2}{\partial x^2} = ?$$ (potrebno za $\langle\hat{p}_x^2\rangle$)

$$G = \frac{\partial F}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial F}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial F}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial F}{\partial\varphi}$$

$$\frac{\partial G}{\partial x} = \frac{\partial^2 F}{\partial x^2} = \frac{\partial r}{\partial x}\frac{\partial G}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial G}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial G}{\partial\varphi}$$

11. Proveriti da li se najverovatnija vrednost ne menja sa vremenom za Θ_r, u stanjima zadatim u zadatku 6.

$$\left| \psi_{n1m_e} \right\rangle, \quad m_e = -1, 0, 1$$

$$\left| \psi_{n1m_e}(t) \right\rangle = \hat{U}(t) \left| \psi_{n1m_e} \right\rangle = e^{-\frac{i}{\hbar} t E_n} \left| \psi_{n1m_e} \right\rangle$$

ili u koord. repr.

$$\psi_{n1m_e}(\vec{r}, t) = e^{-\frac{i}{\hbar} t E_n} \psi_{n1m_e}(\vec{r})$$

$$g_v(\vec{r}, t) = \left| \psi_{n1m_e}(\vec{r}, t) \right|^2 = \left| \psi_{n1m_e}(\vec{r}) \right|^2 = g_v(\vec{r})$$

Dakle, samo gustina verovatnoće ne zavisi od vremena, pa prema tome $\Theta_m \neq \Theta_m(t)$

12. Izračunati matrični element $\langle \Psi | \hat{L}_z + \hat{S}_z | \Psi \rangle$ u stanju $|\Psi\rangle$, zadatom u S_y-reprezentaciji:

a) $Y_1^0 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

b) $Y_1^{-1} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

c) $Y_0^0 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\Psi\rangle = \underbrace{|Y_l^{m_l}\rangle}_{\text{sferni harmonik}} \otimes \underbrace{|U_{m_s}\rangle}_{\text{data matrica}},$$

tj. stanje sistema jeste zadato kao tenzorski proizvod stanja orbitalnih i spinskih stepeni slobode

$$\langle \Psi | \hat{L}_z \otimes \hat{I}_s + \hat{I}_0 \otimes \hat{S}_z | \Psi \rangle =$$

$$\langle Y_l^{m_l} | \otimes \langle U_{m_s} | \hat{L}_z \otimes \hat{I}_s + \hat{I}_0 \otimes \hat{S}_z | Y_l^{m_l} \rangle \otimes | U_{m_s} \rangle =$$

$$\langle Y_l^{m_l} | \hat{L}_z | Y_l^{m_l} \rangle + \langle U_{m_s} | \hat{S}_z | U_{m_s} \rangle$$

S_y - reprezentacija

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad S_z = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

---

a)

$$\langle Y_1^0 | \hat{L}_z | Y_1^0 \rangle = \langle 10 | \hat{L}_z | 10 \rangle = 0$$

$$\langle U_{m_s} | \hat{S}_z | U_{m_s} \rangle = [1\ 0] \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\hbar}{2} [1\ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

Dakle $\langle \psi | \hat{L}_z + \hat{S}_z | \psi \rangle = 0$

b)
c) } Domaći

Za pismeni deo ispita: varijacije na temu!

$\langle \psi | \hat{\vec{L}}^2 + \hat{L}_z | \psi \rangle$

$\langle \psi | \hat{\vec{L}}^2 + \hat{S}_z | \psi \rangle$

13. Izračunati disperziju opservable $\hat{x} \otimes \hat{S}_y$ u stanjima koja su zadata u prethodnom zadatku.

$$\Delta(\hat{x} \otimes \hat{S}_y) = \sqrt{\langle \hat{x}^2 \otimes \hat{S}_y^2 \rangle - \langle \hat{x} \otimes \hat{S}_y \rangle^2}$$

$$\langle \hat{x} \otimes \hat{S}_y \rangle = \langle Y_l^{m_l} | \hat{x} | Y_l^{m_l} \rangle \langle U_{m_s} | \hat{S}_y | U_{m_s} \rangle$$

$$\langle Y_l^{m_l} | \hat{x} | Y_l^{m_l} \rangle = \iiint Y_l^{m_l *}\, r \sin\theta \cos\varphi\, Y_l^{m_l}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Koristiti rekurentne f-le iz Uvoda

$$\langle \hat{x}^2 \otimes \hat{S}_y^2 \rangle = \langle Y_l^{m_l} | \hat{x}^2 | Y_l^{m_l} \rangle \langle U_{m_s} | \hat{S}_y^2 | U_{m_s} \rangle$$

$$\hat{S}_y^2 = \frac{\hbar^2}{4} \hat{\sigma}_y^2 = \frac{\hbar^2}{4} \ldots$$

$$\langle Y_l^{m_l} | \hat{x}^2 | Y_l^{m_l} \rangle = \iiint Y_l^{m_l *} (r \sin\theta \cos\varphi)^2\, Y_l^{m_l}\, \underline{r^2 \sin\theta\, d\theta\, d\varphi}$$

a $\langle U_{m_s} | \hat{S}_y | U_{m_s} \rangle$ na osnovi prethodnog zadatka kao i $\langle U_{m_s} | \hat{S}_y^2 | U_{m_s} \rangle$

↑ Ispit

$$|\psi\rangle \in \underline{\mathcal{H}^{(r)} \otimes \mathcal{H}^{(s)}}$$

↓

nema r

→ Примедба. U pozadi;